Mathematical Finance
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Assignment 13

Deterministic-shift extension (see Brigo and Mercurio [2; 3])[1]

In this exercise, we illustrate a simple method to extend any time-homogeneous short-rate model, so as to exactly reproduce any observed term structure of interest rates while preserving the possible analytical tractability of the original model. We fix some positive integer n and some parameter vector $\alpha \in \mathbb{R}^n$, as well as some $x_0 \in \mathbb{R}$. We consider the following one-dimensional time-homogeneous diffusion

$$x_t^\alpha = x_0 + \int_0^t \mu\big(x_s^\alpha, \alpha\big)\mathrm{d}s + \int_0^t \sigma\big(x_s^\alpha, \alpha\big) \cdot \mathrm{d}W_s^\lambda,\ t \geq 0,$$

where the maps $\mu : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ and $\sigma : \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}^d$ are assumed to be smooth enough so that there is a unique strong solution to the above equation.

The process x^α represents the short-rate in the 'reference model' (that is to say before the extension we will study), and we assume that zero-coupon bond prices can be obtained explicitly in this model, in the sense that there exists some map $z : [0, +\infty)^2 \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that

$$\mathbb{E}^{\mathbb{P}^\lambda}\left[\mathrm{e}^{-\int_t^T x_s^\alpha \mathrm{d}s}\bigg|\mathcal{F}_t\right] =: z\big(t, T, x_t^\alpha, \alpha\big),\ 0 \leq t \leq T.$$

We define now the short-rate as

$$r_t := x_t^\alpha + \varphi(t, \alpha, x_0),\ t \geq 0,$$

where $\varphi : [0, +\infty) \times \mathbb{R}^{n+1} \longrightarrow \mathbb{R}$ is a deterministic function, assumed to be locally integrable. We also give ourselves an initial value $r_0 \in \mathbb{R}$ for the short-rate, and thus enforce

$$\varphi(0, \alpha) + x_0^\alpha = r_0.$$

1) Prove that zero-coupon bond prices in this model are given by

$$B(t, T) = \mathrm{e}^{-\int_t^T \varphi(s,\alpha,x_0)\mathrm{d}s} z\big(t, T, r_t - \varphi(t, \alpha, x_0), \alpha\big),\ 0 \leq t \leq T.$$

2) Define the following reference instantaneous forward rate

$$f_0(T, \alpha) := -\frac{1}{z(0, T, x_0, \alpha)} \frac{\partial z(0, T, x_0, \alpha)}{\partial T}.$$

Prove that the model fits perfectly the currently observed yield curve if and only if

$$\varphi(T, \alpha, x_0) = f^M(0, T) - f_0(T, \alpha),\ T \geq 0,$$

or equivalently

$$\mathrm{e}^{-\int_t^T \varphi(s,\alpha,x_0)\mathrm{d}s} = \frac{B^M(0, T)}{z(0, T, x_0, \alpha)} \frac{z(0, t, x_0, \alpha)}{B^M(0, t)} =: \Phi(t, T, x_0, \alpha).$$

3) Once the initial yield curve has been fitted to the model, how many parameters can we still choose freely in the model. How would you propose to choose them in practice?

4) We now assume that call (and thus put) options on zero-coupon bonds also admit explicit formulae in the reference model. In other words, we suppose that there is some map $Z : [0, +\infty)^3 \times [0, +\infty) \times \mathbb{R} \times \mathbb{R}^n \longrightarrow \mathbb{R}$ such that for any $0 \leq t \leq T \leq s$ and any strike $K \geq 0$

$$\mathbb{E}^{\mathbb{P}^\lambda}\left[\mathrm{e}^{-\int_t^T x_u^\alpha \mathrm{d}u}\big(z\big(T, s, x_T^\alpha, \alpha\big) - K\big)^+\bigg|\mathcal{F}_t\right] = Z\big(t, T, s, K, x_t^\alpha, \alpha\big),\ t \in [0, T].$$

[1] And also Scott [5], Dybvig [4] and Avellaneda and Newman [1] for related approaches.

Prove then that in this model

$$\mathrm{ZBC}_t(T,s,K) = \mathrm{e}^{-\int_t^s \varphi(u,\alpha,x_0)\mathrm{d}u} Z\bigg(t,T,s,K\mathrm{e}^{\int_T^s \varphi(u,\alpha,x_0)\mathrm{d}u}, r_t - \varphi(t,\alpha,x_0),\alpha\bigg).$$

5) Explain, without giving formulae, how we can use the previous question to obtain prices for caps and floors. Give also sufficient conditions, in this model, to be able to obtain explicit formulae for swaptions.

6) We consider the extension of Vašíček's model. meaning that we take $\alpha := (k,\theta,\sigma)$ and

$$x_t^\alpha = x_0 + \int_0^t k\big(\theta - x_s^\alpha\big)\mathrm{d}s + \sigma W_t^\lambda,\ t \geq 0.$$

Compute the function $\varphi(\cdot,\alpha,x_0)$ allowing to fit perfectly the initial yield curve in this case. Show then that we can find a deterministic map η (which you will give explicitly) such that the dynamics of r can be written

$$r_t = r_0 + \int_0^t \big(\eta(s) - kr_s\big)\mathrm{d}s + \sigma W_t^\lambda,\ t \geq 0,$$

and conclude that in this case, the deterministic-shift extension coincides with the Hull–White extended Vašíček model.

7) Answer the same questions when the reference model is the CIR model. Is the extension equivalent to a CIR model with time-dependent coefficients?

Options on futures contracts

We place ourselves again in the context of Chapter 9 in the notes, and we fix throughout the exercise some maturity $T > 0$.

1) For this question (and only this question), we place ourselves in a discrete-time setting where trading can only happen at the fixed instants $(t_i)_{i\in\{0,\dots,n\}}$ for some positive integer n and where

$$0 =: t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n := T.$$

Futures contracts, unlike forward contracts, are marked-to-market, meaning that they receive cash-flows at every trading dates. More precisely, a futures contract is an agreement to purchase an asset at the maturity T, for a pre-specified price, called the *futures price*. This futures price is paid via a sequence of instalments over the contract's life. As with forward contracts, no cash-flow happens at the inception of the contract, supposed to correspond to time 0 here. However, a cash payment is made at every trading date, corresponding to the change in the futures price between this date and the previous trading one. Mathematically, if we define the futures price at time t, for an asset S with maturity T by $G_t(T;S_T)$, then the cash-flows are

$$G_{t_i}(T;S_T) - G_{t_{i-1}}(T;S_T),\ \text{at time } t_i,\ i \in \{1,\dots,n\}.$$

Explain why the value V_t of a futures contract is 0 at any time $(t_i)_{i\in\{0,\dots,n\}}$. Show as well that

$$V_{t_i} = \mathbb{E}^{\mathbb{P}^\lambda}\Bigg[\sum_{k=i+1}^n d(t_i,t_k)\big(G_{t_k}(T;S_T) - G_{t_{k-1}}(T;S_T)\big)\Bigg|\mathcal{F}_{t_i}\Bigg].$$

Prove then that $G_T(T;S_T) = S_T$ and deduce from all the above that the futures prices are actually given by

$$G_{t_i}(T;S_T) = \mathbb{E}^{\mathbb{P}^\lambda}\big[S_T\big|\mathcal{F}_{t_i}\big],\ i \in \{1,\dots,n\}.$$

2) Given the result of the previous question, and coming back to our continuous-time models, we now define the futures price for the tradable asset S, with maturity T, to be

$$G_t(T;S_T) = \mathbb{E}^{\mathbb{P}^\lambda}\big[S_T\big|\mathcal{F}_t\big],\ t \in [0,T].$$

Prove that the difference between forward and futures prices is given by

$$F_t(T;S_T) - G_t(T;S_T) = \frac{\mathbb{C}\mathrm{ov}^{\mathbb{P}^\lambda}[S_T, d(t,T)|\mathcal{F}_t]}{B(t,T)},\ t \in [0,T].$$

Assuming in addition that the asset S has the dynamics

$$S_t = S_0 + \int_0^t S_s\big(r_s \mathrm{d}s + \sigma_s^S \cdot \mathrm{d}W_s^\lambda\big),\ t \in [0,T],$$

and that the process $b(\cdot,T) - \sigma_\cdot^S$ is actually deterministic, then

$$G_t(T;S_T) = F_t(T;S_T)\exp\bigg(\int_t^T \big(b(s,T) - \sigma_s^S\big)\cdot b(s,T)\mathrm{d}s\bigg),\ t \in [0,T],$$

and the dynamics of the futures prices is given by

$$G_t(T;S_T) = G_0(T;S_T) + \int_0^t G_s(T;S_T)\big(\sigma_s^S - b(s,T)\big)\cdot \mathrm{d}W_s^\lambda,\ t \in [0,T].$$

Let us now consider an option written on a futures contract. More precisely, we fix $T \leq s$ and $T \leq u$ and consider the following payoff with maturity T

$$\xi := B(T,s)f\big(G_T(u;S_T)\big).$$

Our goal is now to both replicate and give the price of this option. For this, we will use self-financing portfolios for which we invest in zero-coupon bonds with maturity s and futures contracts on S with maturity u. We also assume that both σ^S and $b(\cdot,s)$ are deterministic functions.

3) If we denote by Δ the process keeping track of how many futures contracts are held in the portfolio, and by Δ^o the one keeping track of how many zero-coupon bonds are held in the portfolio, prove that

$$X_t^{\Delta,\Delta^o} = \Delta_t^o B(t,s),\ X_t^{\Delta,\Delta^o} = X_0^{\Delta,\Delta^o} - \int_0^t \Delta_\ell \mathrm{d}\big(G_\ell(u;S_T)\big) + \int_0^t \Delta_\ell^o \mathrm{d}B(\ell,s),\ t \in [0,T].$$

4) We are looking for replicating self-financing portfolios such that there exists some map $v : [0,T]\times(0,+\infty)^2$, smooth in all its variables, with

$$X_t^{\Delta,\Delta^o} = v\big(t, G_t(u;S_T), B(t,s)\big),\ t \in [0,T].$$

Prove that, necessarily, the map v must satisfy

$$\begin{cases} \partial_t v + \dfrac{x^2}{2}\|\sigma_t^S - b(t,u)\|^2\partial_{xx}^2 v + \dfrac{y^2}{2}\|b(t,s)\|^2\partial_{yy}^2 v + (\sigma_t^S - b(t,u))\cdot b(t,s)xy\partial_{xy}v = 0,\ (t,x,y) \in [0,T)\times(0,+\infty)^2, \\ v(t,x,y) = y\partial_y v(t,x,y),\ (t,x,y)\in[0,T)\times(0,+\infty)^2, \\ v(T,x,y) = yf(x),\ (x,y)\in(0,+\infty)^2. \end{cases}$$

5) By looking for a solution to the previous PDE of the form $v(t,x,y) = yw(t,x)$, and then using Feynman–Kac's formula, prove then that the unique possible value for v is

$$v(t,x,y) = y\int_{\mathbb{R}} g\Big(x\mathrm{e}^{\eta(t,T)+\sqrt{\zeta(t,T)}z - \frac{\zeta(t,T)}{2}}\Big)\frac{\mathrm{e}^{-\frac{z^2}{2}}}{\sqrt{2\pi}},\ (t,x,y)\in[0,T]\times(0,+\infty)^2,$$

where we defined

$$\eta(t,T) := \int_t^T \big(\sigma_\ell^S - b(\ell,u)\big)\cdot b(\ell,s)\mathrm{d}\ell,\ \zeta(t,T) := \int_t^T \big\|\sigma_\ell^S - b(\ell,u)\big\|^2\mathrm{d}\ell,\ t\in[0,T].$$

Give then explicitly the replicating strategy and the no-arbitrage price for the option.

6) We now consider the special case of a call option with maturity T, strike K, written on a futures contract settling at time T for delivery of one unit of S. In other words, we take $T = s = u$ and $g(x) = (x - K)^+$. Give a Black–Scholes–like formula for the value and the replication strategy for this option.

7) Derive similarly the value of a put option with maturity T, strike K, written on a futures contract settling at time T for delivery of one unit of S. Deduce that the call–put parity takes the form

$$C_t\big(T, K; G_T(T; S_T)\big) - P_t t\big(T, K; G_T(T; S_T)\big) = B(t,T)G_t(T; S_T)\mathrm{e}^{\eta(t,T)} - KB(t,T),\ t \in [0,T].$$

What difference can you see compared to call–put parity for options written on forward contracts, or the asset S itself? Comment.

References

[1] M. Avellaneda and J. Newman. Positive interest rates and non–linear term structure models. Technical report, Courant institute of mathematical sciences, New York, 1998.

[2] D. Brigo and F. Mercurio. On deterministic shift extensions of short–rate models. Technical report, Banca IMI, Milan, 1998.

[3] D. Brigo and F. Mercurio. A deterministic–shift extension of analytically–tractable and time–homogeneous short–rate models. *Finance and Stochastics*, 5(3):369–387, 2001.

[4] P.H. Dybvig. Bond and bond option pricing based on the current term structure. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of derivative securities*, pages 271–293. Cambridge University Press, 1997.

[5] L. Scott. The valuation of interest rate derivatives in a multi–factor term structure model with deterministic components. Technical report, University of Georgia, 1995.